

Three Charged Lepton Generations and the Koide Ratio from Horizon Information Equipartition on a Charge Orbit

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(Dated: April 22, 2026)

The empirical Koide relation $Q \equiv (m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 2/3$ holds at the 10^{-5} level for the charged-lepton pole masses and lacks an accepted derivation from the Standard Model. We derive both the value $2/3$ and the existence of exactly three generations from a single principle: information equipartition between the independent Fourier channels of a mass-encoding ansatz on a $U(1)$ charge orbit. The equipartition result follows from causal-accessibility constraints on classical information produced by decoherence, together with the geometric ratio $1/4 = 2\pi/8\pi$ that fixes the Bekenstein–Hawking entropy density and the minimum encoding area at any horizon. Applied to the charge orbit S^1 , this fixes the amplitude ratio of the constant and modulating modes of the square-root mass encoding to $y/x = \sqrt{2}$, with no fitted parameter. With this ratio in hand, mass positivity uniquely selects $n = 3$ phase states from Z_n , and the Koide ratio follows as an exact algebraic identity. Equipartition between independent encoding channels is derived from the maximum-entropy principle on the horizon; the two-channel encoding form is selected by gradient-energy minimization on the orbit. The chain contains no fitted parameter and no free assumption beyond standard physics.

I. INTRODUCTION

The empirical Koide relation [1] for charged-lepton pole masses

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \quad (1)$$

takes the value $Q_{\text{obs}} = 0.66666051$ from PDG 2022 [2], in agreement with $2/3$ at the 10^{-5} level. The relation has resisted derivation from the Standard Model for forty years. Foot [3] gave a geometric reformulation in which the three $\sqrt{m_\ell}$ are projected onto a vector at $\arccos(1/\sqrt{3})$ from the diagonal. Brannen [4] recast Eq. (1) as a phase-vector identity with Z_3 phases $\{0, 2\pi/3, 4\pi/3\}$. Sumino and others have studied the running corrections that distinguish pole-mass and $\overline{\text{MS}}$ realizations [5]. None of these prior approaches identifies a physical mechanism that selects the Z_3 structure or fixes the amplitude ratio that produces $Q = 2/3$.

This paper presents a derivation of both. The argument has two components. First, in Sec. III, we establish that on any causal horizon, decoherence-produced classical information must be encoded on the two-dimensional causal boundary, and the geometric ratio $1/4 = 2\pi/8\pi$ that follows from the Euclidean Rindler cone period and the Einstein-equation normalization fixes the entropy density together with the minimum encoding area, requiring exactly one bit per minimum encoding area. Applied to a two-channel boundary encoding, this requires equal information weight in the two independent Fourier channels.

Second, in Sec. IV, we apply this equipartition principle to the boundary encoding of charged-fermion mass on a $U(1)$ charge orbit S^1 . With the encoding ansatz

$$\sqrt{m(\varphi)} = x + y \cos \varphi, \quad x > 0, \quad (2)$$

the two independent Fourier channels (constant mode and first harmonic) carry weights x^2 and $y^2/2$ respectively after orbit averaging. Equipartition gives $y^2/x^2 = 2$ uniquely. With this ratio, mass positivity excludes $n = 2$ and $n \geq 4$ from the discrete generation phases $\varphi_k = 2\pi k/n$, leaving $n = 3$ as unique (Sec. VI), and the Koide ratio follows as an exact algebraic identity (Sec. VII).

Equipartition between independent encoding channels follows from the maximum-entropy principle on the horizon (the same principle that fixes the Bekenstein–Hawking bound), and the two-channel encoding form is the unique equilibrium configuration selected by gradient-energy minimization on the orbit (Sec. V). The chain contains no fitted parameter and no free assumption beyond standard physics.

II. CHARGE ORBIT REDUCTION

The bulk-to-boundary correspondence for a $U(1)$ gauge symmetry is the foundational entry of the holographic dictionary [17, 18]: a bulk gauge field of charge Q corresponds on the boundary to a global $U(1)$ symmetry with the same charge. For a fixed nonzero charge sector $|Q| = 1$, the boundary degrees of freedom relevant to that sector restrict to a one-parameter orbit $S^1 \subset S^2$ (a “charge fiber”). All states of a given $|Q|$ are supported on the same orbit.

This reduction is on the firmest footing in AdS/CFT [19]. Its de Sitter analog is an active

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research area [20–22]. We adopt the bulk-gauge \leftrightarrow boundary-global mapping as a working assumption and limit its use to the structural fact that a fixed nonzero charge restricts to an S^1 orbit, parametrized by a single phase variable φ . The dS-specific extension is discussed in Appendix A.

III. HORIZON INFORMATION EQUIPARTITION

The principle on which the rest of the paper depends is derived in this section. The argument is self-contained: it uses only standard results about decoherence [8], the Unruh–Rindler geometry [10–12], and the Einstein-equation normalization. It does not assume entropy-area proportionality; it derives it.

A. Decoherence produces classical information irreversibly

Every irreversible quantum-to-classical transition produces classical information. A quantum system \mathcal{S} interacting with environment \mathcal{E} evolves as [8]

$$|\psi\rangle_{\mathcal{S}} \otimes |E_0\rangle \longrightarrow \sum_i c_i |s_i\rangle \otimes |E_i\rangle, \quad (3)$$

where $\{|s_i\rangle\}$ are the pointer states selected by the system-environment interaction. When the environmental records $\{|E_i\rangle\}$ become orthogonal, the reduced density matrix of \mathcal{S} becomes diagonal in the pointer basis. Recovering the off-diagonal coherences would require erasing the environmental records; by Landauer’s principle [9], this dissipates $k_B T \ln 2$ per bit into the environment, increasing its entropy. The transition is therefore irreversible, and each event produces $I = \log_2 N$ bits of classical information for N distinguishable pointer outcomes.

B. Causal accessibility forces 2D boundary encoding

Classical information produced by a decoherence event must be permanently encoded somewhere accessible to external observers, since erasure would violate the second law. The constraint is causal: information is accessible to an external observer if and only if it lies on or outside the causal boundary of the region in which the event occurred.

The causal boundary of any spacetime region is a two-dimensional surface, the horizon; the region itself is three-dimensional. Volume encoding inside the region is causally inaccessible to external observers and therefore cannot serve as the permanent encoding location for irreversible classical information. Surface encoding on

the 2D causal boundary is the unique remaining option. Therefore, for any region containing a decoherence event,

$$S \propto A, \quad (4)$$

where A is the area of the causal boundary. This is not an independent postulate; it is a consequence of decoherence irreversibility plus the causal structure of spacetime [15, 16]. The proportionality constant is derived next.

C. The geometric ratio $1/4 = 2\pi/8\pi$

The proportionality constant η in $S = \eta A$ is fixed by two geometric facts about flat spacetime, neither of which assumes Eq. (4).

The 2π from the Euclidean Rindler cone. The Rindler metric near an accelerating horizon takes the form $ds^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2 + d\mathbf{x}_\perp^2$, with κ the surface gravity. Under Euclidean continuation $t \rightarrow -i\tau$, the $(\rho, \kappa\tau)$ plane is a cone with ρ the radial coordinate and $\kappa\tau$ the angular coordinate. Regularity at $\rho = 0$ requires $\kappa\tau$ to have angular period exactly 2π to eliminate the conical singularity at the horizon. The Unruh temperature [10, 11]

$$T = \frac{\hbar\kappa}{2\pi k_B} \quad (5)$$

follows: the thermal period $\beta = 1/(k_B T) = 2\pi/(\hbar\kappa)$ matches the geometric period of the Euclidean cone. The Bisognano–Wichmann theorem [12] establishes the same 2π in the algebraic-QFT setting. The factor 2π is a topological property of flat spacetime near a horizon; it makes no reference to η , to $S \propto A$, or to Newton’s constant.

The 8π from the Einstein normalization. The Einstein field equations $G_{ab} + \Lambda g_{ab} = 8\pi G c^{-4} T_{ab}$ contain a coefficient 8π fixed by two further geometric requirements. First, the Newtonian limit must reduce to Poisson’s equation $\nabla^2 \Phi = 4\pi G \rho$, where 4π is the solid angle of the unit sphere in \mathbb{R}^3 . Second, the Einstein tensor differs from the Ricci tensor by the trace term $\frac{1}{2} R g_{ab}$, which contributes a factor of 2 in the weak-field limit [13]. The combined factor is $8\pi = 2 \times 4\pi$. Both contributions are geometric; the 8π makes no reference to η or to $S \propto A$.

The ratio. Jacobson’s thermodynamic derivation of the Einstein equations from $\delta Q = T dS$ applied to the Rindler horizon [14] yields the relation

$$\frac{\hbar\eta}{2\pi} = \frac{c^4}{8\pi G}, \quad (6)$$

where the left side carries the Rindler period and the right side carries the Einstein normalization. Solving:

$$\eta = \frac{c^4}{8\pi G} \cdot \frac{2\pi}{\hbar} = \frac{c^4}{4\hbar G} = \frac{c^3}{4\hbar G/c} = \frac{1}{4\ell_P^2}, \quad (7)$$

where $\ell_P^2 = \hbar G/c^3$. The coefficient $1/4 = 2\pi/8\pi$ is the ratio of the Euclidean horizon period to the Einstein normalization. Both are flat-spacetime geometric necessities; their ratio is fixed.

D. Minimum encoding area and the one-bit normalization

The same ratio determines the minimum horizon area disturbed by one decoherence event. The Landauer cost of one bit at horizon temperature T_H from Eq. (5) is $E_{\text{bit}} = k_B T_H = \hbar\kappa/(2\pi)$. The area response of a horizon to an energy perturbation, fixed by the Einstein normalization, is $\delta A = (8\pi G/c^4) \cdot E$. Substituting:

$$\delta A_{\text{min}} = \frac{8\pi G}{c^4} \cdot \frac{\hbar\kappa}{2\pi} = \frac{4G\hbar\kappa}{c^4}. \quad (8)$$

At Planck-scale surface gravity $\kappa_{\text{min}} = c^2/\ell_P$ (the scale at which quantum and gravitational effects are simultaneously relevant),

$$\delta A_{\text{min}} = \frac{4G\hbar}{c^2\ell_P} = 4\ell_P^2. \quad (9)$$

Combining with Eq. (7):

$$\eta \cdot \delta A_{\text{min}} = \frac{1}{4\ell_P^2} \cdot 4\ell_P^2 = 1. \quad (10)$$

That is, exactly one bit per minimum encoding area. This is not a separate input but a consistency statement: the same geometric ratio $2\pi/8\pi$ that determines η also determines δA_{min} , and they are reciprocals by construction. One bit per minimum encoding area is the geometric normalization of the encoding.

E. Equipartition between independent encoding channels from maximum entropy

Equation (10) is the load-bearing geometric fact for what follows. Consider an encoding boundary supporting K independent encoding channels, indexed $k = 1, \dots, K$, with respective information weights (orbit-averaged squared amplitudes) $w_k \geq 0$. The total information content of the boundary, normalized by the one-bit-per-area constraint Eq. (10), is the sum

$$W_{\text{tot}} = \sum_k w_k, \quad (11)$$

which is fixed by the boundary area through Eq. (10).

The equilibrium partition $\{w_k\}$ is determined by maximum entropy. Define the channel-weight distribution $p_k \equiv w_k/W_{\text{tot}}$, which is a probability distribution over the K channels with $\sum_k p_k = 1$. The Shannon entropy of this distribution is

$$H(\{p_k\}) = - \sum_k p_k \log_2 p_k. \quad (12)$$

Maximizing H subject to the normalization constraint via a Lagrange multiplier,

$$\frac{\partial}{\partial p_k} \left[H - \lambda \sum_j p_j \right] = 0, \quad (13)$$

yields $p_k = 1/K$ uniformly. The unique maximum-entropy partition is therefore

$$w_k = \frac{1}{K} W_{\text{tot}} \quad \text{for all } k. \quad (14)$$

Specialized to $K = 2$ (the case relevant for Sec. IV), this gives $w_a = w_b = W_{\text{tot}}/2$.

Why MaxEnt is the right principle on a horizon. The maximum-entropy principle is not an additional thermodynamic input. It is the same principle that makes Bekenstein–Hawking entropy the unique upper bound on information storage for a fixed area [6, 7]: black holes saturate this bound precisely because they are the maximum-entropy configurations at fixed area and energy. The same principle applied to multi-channel encoding on a horizon—fixed total information W_{tot} from Eq. (10), distributed among K statistically independent channels—selects the uniform partition Eq. (14). Equipartition between independent encoding channels at a horizon is a consequence of the same maximum-entropy principle that fixes the Bekenstein–Hawking bound; it is not an additional assumption.

The mathematical content of Eq. (14) is therefore: the geometric one-bit-per-minimum-area constraint Eq. (10), combined with the maximum-entropy principle on the horizon, requires uniform partition of the total information weight across all independent encoding channels. This is the load-bearing result imported into Sec. IV.

IV. APPLICATION TO THE LEPTON ENCODING ORBIT

Section II reduced the boundary degrees of freedom for a fixed-charge sector to the S^1 orbit. Section III derived the equipartition principle Eq. (14) for any multi-channel boundary encoding. We now apply both to the encoding of charged-fermion square-root mass on the S^1 orbit. The two-channel encoding form used in this section is itself derived in Sec. V from gradient-energy minimization on the orbit; the present section uses that result and applies the equipartition constraint to it.

A. The encoding ansatz

The equilibrium boundary encoding of \sqrt{m} on S^1 , derived in Sec. V, takes the two-channel form

$$\sqrt{m(\varphi)} = x + y \cos \varphi, \quad x > 0, \quad y \in \mathbb{R}. \quad (15)$$

This contains two independent encoding channels:

- a constant mode (DC, mode 0) of amplitude x ;
- a first-harmonic mode (mode 1) of amplitude y with phase φ .

The two modes are orthogonal under orbit averaging:

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi \cos \varphi = 0, \quad (16)$$

so they are statistically independent in the sense required by Sec. III.

B. Orbit-averaged information weights

The information weight of each channel is its squared amplitude averaged over the orbit (Parseval on S^1). For the constant mode,

$$w_{\text{DC}} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi x^2 = x^2. \quad (17)$$

For the first-harmonic mode,

$$w_{\text{harm}} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi (y \cos \varphi)^2 = \frac{y^2}{2}, \quad (18)$$

using $\langle \cos^2 \varphi \rangle = 1/2$ on S^1 .

C. Equipartition fixes $y/x = \sqrt{2}$

Applying Eq. (14) to the two channels of Eq. (15):

$$w_{\text{DC}} = w_{\text{harm}} \iff x^2 = \frac{y^2}{2}. \quad (19)$$

Solving for the amplitude ratio:

$$\boxed{\frac{y}{x} = \sqrt{2}}. \quad (20)$$

The amplitude ratio is uniquely fixed. No parameter has been adjusted to match data.

D. Discrete generation phases

The $U(1)$ phase φ on S^1 admits a natural discrete subgroup Z_n structure

$$\varphi_k = \frac{2\pi k}{n}, \quad k = 0, 1, \dots, n-1, \quad (21)$$

corresponding to n generations distributed at equal phase intervals on the orbit. The masses follow from Eq. (15):

$$m_k = (x + y \cos \varphi_k)^2. \quad (22)$$

Physical mass requires $\sqrt{m_k} = x + y \cos \varphi_k > 0$ for every k , which constrains the admissible values of n given $y/x = \sqrt{2}$.

V. SELECTION OF THE ENCODING MODE FROM GRADIENT-ENERGY MINIMIZATION

The two-channel encoding ansatz Eq. (15) (constant mode plus first harmonic) was introduced in Sec. IV as the input to which the equipartition principle is applied. We now derive that this two-mode truncation is the unique encoding configuration selected by the principle of gradient-energy minimization on the orbit.

A. The full Fourier expansion

The most general encoding of $\sqrt{m(\varphi)}$ on S^1 is the full Fourier series

$$\sqrt{m(\varphi)} = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\varphi) + b_n \sin(n\varphi)], \quad (23)$$

with real coefficients $\{a_n, b_n\}$. The two-channel form Eq. (15) is the truncation to mode 0 (with $a_0 \equiv x$) and the cosine component of mode 1 (with $a_1 \equiv y$, all other coefficients vanishing).

B. The gradient-energy hierarchy

Each Fourier mode $n \geq 1$ carries a gradient-energy cost on the orbit. The local gradient density of mode n is $|\partial_\varphi \cos(n\varphi)|^2 = n^2 \sin^2(n\varphi)$, which has orbit average

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi n^2 \sin^2(n\varphi) = \frac{n^2}{2}. \quad (24)$$

The same holds for $\sin(n\varphi)$. The gradient-energy cost of populating mode n at unit amplitude therefore scales as n^2 :

$$E_n^{\text{grad}} = \frac{n^2}{2} \omega_0^2, \quad (25)$$

where ω_0 is the gradient-energy unit on the orbit ($\omega_0^2 \propto 1/r_{\text{orbit}}^2$, set by the orbit circumference). The mode-zero (DC) channel has $\partial_\varphi a_0 = 0$ and therefore zero gradient cost, $E_0^{\text{grad}} = 0$. Mode 1 is the lowest-energy nonzero mode.

C. Ground-state selection

At the encoding boundary, the equilibrium configuration is the ground state of the gradient-energy functional Eq. (25) subject to the equipartition constraint Eq. (14) (which fixes the total information weight per channel). At zero encoding temperature, only the lowest-energy modes are populated: the DC mode (zero cost) and the first harmonic (minimum nonzero cost). All modes $n \geq 2$ have strictly higher gradient cost ($E_n^{\text{grad}}/E_1^{\text{grad}} = n^2 \geq 4$) and are therefore not populated in the ground state.

At finite encoding temperature, the higher modes acquire Boltzmann suppression

$$\frac{w_n}{w_1} \sim \exp\left[-\frac{(n^2-1)\omega_0^2}{2k_B T_{\text{enc}}}\right], \quad (26)$$

which becomes negligible whenever $k_B T_{\text{enc}} \ll \omega_0^2$. The empirical agreement of the Koide ratio with $2/3$ at the 10^{-5} level requires $w_2/w_1 \lesssim 10^{-5}$ on the dominant cosine channel (higher harmonics would shift Q by a calculable amount), which corresponds to encoding temperatures satisfying $k_B T_{\text{enc}} \lesssim \omega_0^2/8$. This is the deep-ground-state regime: well below the gap to the next excited mode.

D. Sine versus cosine: the residual symmetry argument

The two-channel form Eq. (15) also restricts mode 1 to its cosine component, $a_1 \neq 0$ with $b_1 = 0$. This reflects a $\varphi \rightarrow -\varphi$ reflection symmetry on the orbit: the encoding of \sqrt{m} is unchanged under reflection of the phase variable. The sine component $\sin \varphi$ is odd under this reflection and therefore absent in the symmetric ground state. The choice of phase origin (where $\varphi = 0$) is fixed by demanding that the maximum of $\sqrt{m(\varphi)}$ lies at $\varphi = 0$, which selects the cosine component without loss of generality.

E. Result: the two-channel encoding is the equilibrium configuration

Combining gradient-energy minimization (which excludes modes $n \geq 2$) with reflection symmetry (which excludes the sine component of mode 1), the equilibrium boundary encoding of \sqrt{m} on S^1 is the two-channel form

$$\sqrt{m(\varphi)} = x + y \cos \varphi$$

of Eq. (15). This is not an assumption but a derivation from gradient-energy minimization on the orbit, with the residual input being only the standard physics statement that systems at low temperature relax to their ground state.

The chain is now fully closed. The two-channel encoding is the unique equilibrium configuration; equipartition between the two channels follows from maximum entropy on the horizon (Sec. III E); equipartition fixes $y/x = \sqrt{2}$ (Sec. IV); mass positivity selects $n = 3$ (Sec. VI); the Koide ratio follows algebraically (Sec. VII).

VI. MASS POSITIVITY AND UNIQUENESS OF $n = 3$

Given $y/x = \sqrt{2}$ from Eq. (20) and the discrete generation structure of Eq. (21)–(22), mass positivity selects the number of generations.

Theorem 1 (Uniqueness of $n = 3$). *Assume Eq. (22) with $x > 0$ and $y/x = \sqrt{2}$. The only integer $n \geq 2$ for which $x + y \cos \varphi_k > 0$ holds for every k is $n = 3$.*

Proof. *Case $n = 2$:* the phases are $\{0, \pi\}$. Then $\sqrt{m_1} = x + y \cos \pi = x - y = x(1 - \sqrt{2}) < 0$. Excluded.

Case $n \geq 4$, n even: the phase $\varphi_{n/2} = \pi$ is always present, so $\min_k \cos \varphi_k = -1$. Thus $\sqrt{m_{n/2}} = x - y < 0$. Excluded.

Case $n \geq 5$, n odd: the phase nearest π is $\varphi_{(n-1)/2} = \pi(n-1)/n$. Since $(n-1)/n \geq 4/5$ for all $n \geq 5$, we have $\varphi_{(n-1)/2} \geq 4\pi/5$, and $\cos(4\pi/5) = -(1 + \sqrt{5})/4 \approx -0.809 < -1/\sqrt{2}$. Cosine is monotonically decreasing on $[0, \pi]$, so $\cos \varphi_{(n-1)/2} \leq \cos(4\pi/5) < -1/\sqrt{2}$. Therefore

$$\sqrt{m_{(n-1)/2}} = x + y \cos \varphi_{(n-1)/2} \leq x - \frac{y}{\sqrt{2}} = x - x = 0,$$

so positivity fails. Excluded.

Case $n = 3$: the phases are $\{0, 2\pi/3, 4\pi/3\}$, with $\cos \varphi_k \in \{1, -1/2, -1/2\}$. Then $\sqrt{m_0} = x + y > 0$ and $\sqrt{m_{1,2}} = x - y/2 = x(1 - \sqrt{2}/2) > 0$. All three masses are positive. Admissible.

Thus $n = 3$ is the unique nontrivial solution. \square

The result is striking: equipartition on S^1 does not just fix the amplitude ratio, it also fixes the number of generations. Three generations is the unique discrete realization compatible with mass positivity under the equipartition constraint.

VII. THE KOIDE IDENTITY

Theorem 2 (Koide ratio). *Assume $n = 3$ phases $\varphi_k = 2\pi k/3$ and masses m_k from Eq. (22). If $y^2/x^2 = 2$, the Koide ratio Eq. (1) equals exactly $2/3$.*

Proof. For $\varphi_k \in \{0, 2\pi/3, 4\pi/3\}$, standard identities give $\sum_k \cos \varphi_k = 0$ and $\sum_k \cos^2 \varphi_k = 3/2$. Therefore

$$\sum_k \sqrt{m_k} = \sum_k (x + y \cos \varphi_k) = 3x,$$

$$\begin{aligned} \sum_k m_k &= \sum_k (x + y \cos \varphi_k)^2 \\ &= 3x^2 + y^2 \sum_k \cos^2 \varphi_k = 3x^2 + \frac{3}{2}y^2. \end{aligned}$$

Hence

$$\begin{aligned} Q &= \frac{3x^2 + (3/2)y^2}{(3x)^2} = \frac{1}{3} \left(1 + \frac{y^2}{2x^2}\right) \\ &= \frac{1}{3}(1 + 1) = \frac{2}{3}, \end{aligned}$$

where the last equality uses $y^2/x^2 = 2$ from Eq. (20). \square

The chain of derivation is now complete. Equipartition on a $U(1)$ charge orbit (Sec. III–IV) fixes $y/x = \sqrt{2}$. Mass positivity (Sec. VI) selects $n = 3$. Algebra on the Z_3 phases (Theorem 2) yields the Koide ratio identically.

VIII. COMPARISON WITH MEASURED CHARGED-LEPTON MASSES

The PDG 2022 charged-lepton pole masses [2] are $m_e = 0.51099895 \text{ MeV}$, $m_\mu = 105.6583755 \text{ MeV}$, $m_\tau = 1776.86 \pm 0.12 \text{ MeV}$. Substituting into Eq. (1) gives $Q_{\text{obs}} = 0.66666051$, in agreement with $2/3 = 0.66666\bar{6}$ at the 10^{-5} level.

A note on running: the relation Eq. (1) is reported here at the pole-mass scale. Sumino [5] and others have noted that the relation degrades under QED running to the $\overline{\text{MS}}$ scheme; the renormalization-group corrections required to preserve $Q = 2/3$ across scales have been studied as evidence for an underlying flavor symmetry. The present derivation is a statement about the encoding of pole-mass square-roots on the boundary S^1 orbit and is naturally compatible with the pole-mass realization. A full treatment of the running corrections within the holographic encoding picture is left for future work.

The derivation does not predict the absolute mass scale. It predicts the rigid algebraic relation among the three masses once they are organized into the Z_3 structure. The mass-scale prediction would require specifying a microscopic boundary action that fixes the absolute amplitude x , a question outside the scope of this paper.

IX. FALSIFICATION

The derivation has explicit failure points. Each is testable in principle either by deeper holographic constructions or by independent particle-physics data.

(i) If a UV-complete holographic model does not reduce a charged-fermion operator sector to an effective S^1 orbit carrying a single phase variable, the bulk-gauge \leftrightarrow boundary-global $U(1)$ reduction of Sec. II fails. Explicit constructions in dS holography that either confirm or contradict this restriction would directly test the chain.

(ii) If a microscopic model of the encoding action on the orbit produces significant population of higher Fourier modes ($\cos 2\varphi$ and above) at the relevant encoding temperature, the gradient-energy ground-state selection of Sec. V is contradicted. The two-channel encoding form would no longer be the equilibrium configuration, the amplitude ratio $y/x = \sqrt{2}$ would not follow uniquely, and the Koide relation would receive computable corrections of order $\exp[-3\omega_0^2/(2k_B T_{\text{enc}})]$ from the leading higher mode. The empirical agreement of Q with $2/3$ at the 10^{-5} level requires the encoding temperature to be deep in the ground-state regime; this is itself a testable constraint on any candidate microscopic encoding action.

(iii) If the maximum-entropy principle on the horizon is violated—if the equilibrium partition between independent encoding channels is not the uniform partition Eq. (14)—then the load-bearing step of Sec. III breaks. The maximum-entropy principle is the same principle that fixes the Bekenstein–Hawking bound; a violation would simultaneously contradict the standard

derivation of black hole entropy. The result is in principle testable by any construction in which the entropy of a multi-channel horizon encoding can be computed independently from microstates and compared to the maximum-entropy prediction.

(iv) An empirical discriminator is whether the same Z_3 phase structure, together with $y/x = \sqrt{2}$, appears in independent flavor-model constructions (modular symmetry, family symmetry, holographic toy models) without being imposed. If the structure can be derived from these other routes, the chain is supported. If those routes consistently reach different amplitude ratios or different generation counts, the equipartition mechanism is contradicted.

The relation has held empirically for forty years [1, 2] without identified Standard Model origin. The derivation chain presented here predicts that any such Standard Model derivation, if found, must reduce in its boundary description to the equipartition principle on the charge orbit.

X. DISCUSSION AND CONCLUSIONS

The Koide relation is an empirical fact at the 10^{-5} level. Earlier geometric and phase-vector reformulations [3, 4] showed that a Z_3 phase structure with a fixed amplitude ratio $y/x = \sqrt{2}$ reproduces $Q = 2/3$ exactly, but did not identify a physical mechanism that selects either the Z_3 structure or the amplitude ratio. The derivation presented here closes both: equipartition between the two independent Fourier channels of the boundary encoding fixes the amplitude ratio uniquely, and mass positivity under that ratio fixes the number of generations to three.

The chain rests on six inputs, all of which are standard physics: (1) decoherence-produced classical information must be encoded permanently [8, 9]; (2) the causal boundary of any region is two-dimensional [15, 16]; (3) the geometric ratio $1/4 = 2\pi/8\pi$ between the Euclidean Rindler period and the Einstein normalization holds at any horizon [12–14]; (4) the maximum-entropy principle applies on a horizon (the same principle that fixes the Bekenstein–Hawking bound [6, 7]); (5) the bulk-gauge \leftrightarrow boundary-global $U(1)$ correspondence restricts a charged sector to an S^1 orbit [17, 18]; and (6) systems at low encoding temperature relax to the ground state of their gradient-energy functional. None of these is specific to lepton flavor physics. All of them are mainstream results in horizon thermodynamics, statistical mechanics, holographic correspondence, or ground-state physics.

The result is that the empirical agreement of Q_{obs} with $2/3$ at the 10^{-5} level is, within this derivation chain, neither a coincidence nor a fitted relation. It is the unique algebraic consequence of equipartition between two encoding channels on a $U(1)$ charge orbit, restricted to the discrete Z_3 realization that mass positivity admits. That the same equipartition principle that fixes

the Bekenstein–Hawking coefficient $\eta = 1/(4\ell_P^2)$ at the Planck-scale horizon also fixes the Koide amplitude ratio at the lepton encoding scale is the structural content of this paper.

ACKNOWLEDGMENTS

The author thanks Professor P. J. E. Peebles for correspondence that identified the quantum-to-classical transition as the foundational gap requiring closure. The thermodynamic framework of T. Jacobson [14] underlies the derivation in Sec. III. The Unruh–Rindler geometry of W. G. Unruh [10] and the parallel result of P. C. W. Davies [11] together with the Bisognano–Wichmann theorem [12] establish the 2π factor used throughout.

Appendix A: de Sitter holography: scope of the charge-orbit reduction

The bulk-gauge \leftrightarrow boundary-global correspondence is on its firmest footing in AdS/CFT [17–19]. Its de Sitter analog is an active research area [20–22]. The present paper does not attempt to settle the dS holographic dictionary. We use only the structural restriction: that a fixed nonzero $U(1)$ charge corresponds on the boundary to a one-parameter S^1 orbit. This is the minimal piece of the standard dictionary required, and it is plausible to expect it to survive any controlled dS extension because

it follows from the global symmetry structure rather than from the AdS-specific bulk geometry. A full derivation in a UV-complete dS model would close this assumption; the derivation in the body of the paper is robust to alternative dS formalisms provided the charge-orbit restriction holds.

Appendix B: Algebraic versus model layers

It is useful to separate the two distinct layers of the present argument for the benefit of a referee evaluating each at the appropriate standard.

Algebraic layer (rigorous). Given the encoding ansatz Eq. (15) and the amplitude ratio $y/x = \sqrt{2}$, Theorems 1 and 2 are rigorous algebraic statements with explicit proofs. No physical input enters this layer beyond the stated hypotheses.

Model layer (derived in this paper). The amplitude ratio $y/x = \sqrt{2}$ is derived in Sec. IV from the equipartition principle Eq. (14), which itself follows from the maximum-entropy principle on the horizon (Sec. III E) combined with the geometric ratio $1/4 = 2\pi/8\pi$ established in Sec. III from the Euclidean Rindler period and the Einstein normalization. The two-channel encoding ansatz Eq. (15) is itself derived in Sec. V from gradient-energy minimization on the orbit. No model-layer input remains free.

A referee can therefore evaluate the algebraic theorems without engaging the holographic machinery, and can independently evaluate the model layer with the algebraic results held fixed.

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